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NONLINEAR THEORY OF PLASMA OSCILLATIONS

by

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ABSTRACT

The nonlinear properties of electrostatic waves in a uniform, zero-temperature plasma free from magnetic field is investigated by two independent perturbation procedures. The first of these is the "derivative-expansion technique," applied to the Lagrangian-variable formalism. The second is a canonical-transformation procedure based on a Hamiltonian description. Both procedures lead to the same formula for the dominant (four-wave) interaction process.

If the spectrum is one-dimensional, the wave interaction vanishes. In general, the effect of wave interactions may be divided into "coherent" and "incoherent" contributions. The former leads to a frequency displacement which may be characterized by a dispersion relation. This is evaluated for a test wave in a thermally excited plasma and found to be of the standard form. The study of incoherent interaction is postponed for a subsequent article.

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I. INTRODUCTION

The great majority of published work on the theory of waves in plasmas is restricted to the linear approximation. One reason for investigating consequences of the nonlinear nature of the equations describing waves in a plasma is, therefore, to determine the conditions under which the linear approximation may be considered satisfactory. Another and more appealing reason is to determine what phenomena involving waves in plasmas are due essentially to nonlinearity of the wave equations. A review of contributions to the nonlinear theory of plasma waves in general has recently appeared.[1] The present article will be concerned with nonlinear aspects of electrostatic waves, otherwise termed "plasma oscillations."

Most earlier work on this subject has taken advantage of the considerable simplification to be obtained by adopting a one-dimensional model for the excited plasma. Dawson[2] has shown that the motion of a cold single-stream electron plasma may be solved for arbitrary initial conditions under the one-dimensional restriction, provided the amplitude is not so large as to produce cross-over of electron trajectories. Sen,[3] Smerd[4] and Gould[5] have found traveling-wave solutions of the nonlinear equations describing electrostatic waves in two interpenetrating zero-temperature electron streams. Bernstein, Green and Kruskal[6] have derived similar traveling-wave solutions of the equations governing electrostatic waves in a plasma of arbitrary electron-velocity distribution function. Buneman[7] has investigated, by numerical calculation, the development into the nonlinear regime of two-stream electrostatic instability. Drummond and Pines[8] have recently studied, within a formalism which is not restricted to one-dimensional models, the one-dimensional nonlinear development of weak electrostatic instability in a plasma of nonzero temperature.

An analysis of the nonlinear behavior of electrostatic waves in a zero-temperature plasma, which was not subject to the one-dimensional restriction, was published by the author[9] in 1957; this article will be referred to as I. This theory was developed within the Eulerian framework and used a perturbation procedure for approximate evaluation of nonlinear effects which is here termed the "derivative-expansion" technique. This procedure is related to the Kryloff-Bogoliuboff

technique[10] for treating nonlinear dynamical systems, and also to the Chapman-Enskog technique[11] for analysis of nonuniform gases. The procedure is also analogous to the expansion of interaction processes in a gas into binary collisions, three particle collisions, etc. Calculations were carried through to the lowest significant order -- that of four-wave interaction (corresponding to binary collision). The key equation of I is (6.7) together with formula (7.1), giving the rate of change of the amplitude of a wave due to interaction with three other waves. From this formula, one may proceed to calculate physical processes such as dispersion and spectral decay. The corresponding equation of the present article is (4.13), and the corresponding formula (4.19).

It was asserted in Section 5 of I that a one-dimensional spectrum exhibits dispersion. This assertion is incorrect, and the article has been criticized accordingly;[12] however, this error was due to an algebraic mistake, as has been pointed out by Jackson[13] or as may be verified directly from formula (7.1) of I.

However, the treatment of I is open to criticism on other grounds. If one considers the limiting form of the kernel as $\mathbf{k}_1 \to \mathbf{k}_3$ (and, in consequence, $\mathbf{k}_2 \to \mathbf{k}_4$), one finds that the limiting value of the kernel depends upon the manner in which this limiting process is effected. This property of the kernel is physically unreasonable and must be due to one of the following possibilities: (i) an algebraic error; (ii) a fundamental shortcoming of the mathematical procedure adopted; or (iii) a minor shortcoming of the mathematical procedure such as an improperly chosen subsidiary condition. In this article and a subsequent article by Ball and Sturrock, we shall show that the error was due to the third of the above possibilities and we shall derive the correct form of the kernel. In the present article, the same problem is attacked by two methods distinct from each other and from that originally adopted: first by applying the derivative-expansion technique to the problem posed in terms of Lagrangian variables, and second by applying the theory of canonical transformations to a Hamiltonian formulation of the problem. These two methods yield an identical kernel which does not suffer from the objection raised against that of I. In the following article, it will be shown that a subsidiary condition in the original treatment was improperly chosen and that, if this error is corrected, the original theory is in agreement with that now presented.

As shown in I, wave-interaction may be divided into two classes: "coherent," which give rise to a frequency shift and may be characterized by a dispersion relation, and "incoherent," which give rise to spectral decay. In this article we shall consider only the former category and postpone investigation of incoherent effects for a subsequent article.

II. CONSTRUCTION OF THE LAGRANGIAN AND HAMILTONIAN FUNCTIONS

The model which we consider is that of a uniform distribution of electrons, of density n, neutralized by an equal density of ions which we assume to be stationary. We describe the motion of electron gas by a displacement-vector $\xi_r(x,t)$, where x is used as an abbreviation for (x_1, x_2, x_3) and $r (= 1, \text{etc.})$ takes the values 1, 2, 3; and $x_r + \xi_r(\underline{x},t)$ is the position at time t of the electron which, in the quiescent state of the electron gas, is at position x_r. The quiescent state is that in which electrons are uniformly distributed and are at rest. If we describe the electrostatic interaction between electrons by the Green function $G(\underline{x})$,

$$G(\underline{x}) = \frac{1}{|\underline{x}|} \quad ,$$

the motion of the electron gas is given by the action principle

$$\delta S = 0 \quad ,$$

where

$$S = \int d^3x \, \frac{1}{2} nm \left(\frac{\partial \xi_r}{\partial t}\right)^2 - \frac{1}{2} e^2 n^2 \iint d^3x \, d^3x'$$

$$\cdot \left[G\left(\underline{x} + \underline{\xi}(\underline{x},t) - \underline{x}' - \underline{\xi}(\underline{x}',t)\right) - G\left(\underline{x} + \xi(\underline{x},t) - \underline{x}'\right)\right] .$$

To simplify calculations, we assume that the excitation is periodic with respect to a large cubical unit cell of volume V. One may show that, if the disturbance extends over a finite region of space, the interaction between disturbances in different cells may be made negligibly small by making the dimensions of the cells sufficiently large. Hence we may consider the integrations in (...) to be taken over a unit cell.

We now proceed to Fourier-analyze, writing

$$\phi(\underline{x}) = \sum \phi(\underline{k}) e^{i\underline{k}\cdot\underline{x}} \qquad (2.4)$$

and

$$\phi(\underline{k}) = V^{-1} \int d^3x \, \phi(\underline{x}) e^{-i\underline{k}\cdot\underline{x}} \quad . \qquad (2.5)$$

The summation in (2.4) extends over all values of $\underline{k}$ for which $e^{i\underline{k}\cdot\underline{x}}$ has the same value over the surface of the unit cube. We note that

$$G(\underline{k}) = \frac{4\pi V^{-1}}{k^2} \quad . \qquad (2.6)$$

We now analyze S as follows:

$$S = S^{(0)} + S^{(1)} + S^{(2)} + \dots \quad , \qquad (2.7)$$

where $S^{(n)}$ is of the n'th order in the dynamical variables $\xi_r(\underline{x},t)$. Clearly, $S^{(0)}$ will not contribute to the equations of motion. Since $\xi_r \equiv 0$ is a solution of the equations of motion, we also expect that $S^{(1)}$ may be ignored, and it may be verified by calculation that $S^{(1)} = 0$. We obtain, for the next three terms, the following formulas:

$$S^{(2)} = \frac{1}{2} nm \int d^3x \left(\frac{\partial \xi_r}{\partial t}\right)^2 + \frac{1}{2} e^2 n^2 \iint d^3x \, d^3x' \xi_r(\underline{x}) \xi_r(\underline{x}') \frac{\partial^2 G(\underline{x} - \underline{x}')}{\partial x_r \partial x_s} , \qquad (2.8)$$

$$S^{(3)} = \frac{1}{2} e^2 n^2 \iint d^3x \, d^3x' \xi_r(\underline{x}) \xi_s(\underline{x}) \xi_t(\underline{x}') \frac{\partial^3 G(\underline{x} - \underline{x}')}{\partial x_r \partial x_s \partial x_t} \quad . \qquad (2.9)$$

$$S^{(4)} = e^2 n^2 \iint d^3x \, d^3x' \left[\frac{1}{6} \xi_r(\underline{x}) \xi_s(\underline{x}) \xi_t(\underline{x}) \xi_u(\underline{x}') - \frac{1}{8} \xi_r(\underline{x}) \xi_s(\underline{x}) \xi_t(\underline{x}') \xi_u(\underline{x}') \right] \times \frac{\partial^4 G(\underline{x} - \underline{x}')}{\partial x_r \partial x_s \partial x_t \partial x_u} \quad . \qquad (2.10)$$

where we adopt the summation convention and, for brevity, suppress the argument t of $\xi_r(\underline{x},t)$.

We now Fourier-transform according to the scheme (2.4), (2.5), and use (2.6). The Lagrangian function so obtained is related to the action integral by

$$S = nmV.L \quad . \tag{2.11}$$

The quadratic, cubic and quartic terms of L are found to be

$$L^{(2)} = \frac{1}{2} \sum_{\underline{k}+\underline{k}'=0} \left\{ \dot{\xi}_r(\underline{k})\dot{\xi}_r(\underline{k}') - \omega_p^2 \frac{1}{\underline{k}^2} \underline{k}.\xi(\underline{k})\underline{k}.\xi(\underline{k}') \right\} , \tag{2.12}$$

$$L^{(3)} = -\frac{1}{2} i\omega_p^2 \sum_{\underline{k}+\underline{k}'+\underline{k}''=0} \left(\frac{1}{\underline{k}^2}\right) \underline{k}.\underline{\xi}(\underline{k})\underline{k}.\underline{\xi}(\underline{k}')\underline{k}.\underline{\xi}(\underline{k}'') \quad , \tag{2.13}$$

$$L^{(4)} = \omega_p^2 \sum_{\underline{k}+\underline{k}'+\underline{k}''+\underline{k}'''=0} \left\{ \frac{1}{6} \left(\frac{1}{\underline{k}^2}\right) \underline{k}.\underline{\xi}(\underline{k})\underline{k}.\underline{\xi}(\underline{k}')\underline{k}.\underline{\xi}(\underline{k}'')\underline{k}.\underline{\xi}(\underline{k}''') \right.$$

$$\left. - \frac{1}{8} \left(\frac{1}{(\underline{k}+\underline{k}')^2}\right) (\underline{k}+\underline{k}').\underline{\xi}(\underline{k})(\underline{k}+\underline{k}').\underline{\xi}(\underline{k}')(\underline{k}+\underline{k}').\underline{\xi}(\underline{k}'')(\underline{k}+\underline{k}').\underline{\xi}(\underline{k}''') \right\} . \tag{2.14}$$

We introduce the momentum variable cononically conjugate to $\xi_r(\underline{k},t)$ as follows:

$$\pi_r(\underline{k},t) = \frac{\partial L}{\partial \dot{\xi}_r(\underline{k},t)} = \dot{\xi}_r(-\underline{k},t) \quad . \tag{2.15}$$

We now form the Hamiltonian function

$$H = \sum_{\underline{k}} \pi_r(\underline{k})\dot{\xi}_r(\underline{k}) - L \quad , \tag{2.16}$$

and find that

$$H^{(2)} = \frac{1}{2} \sum_{\underline{k}+\underline{k}'=0} \left\{ \pi_r(\underline{k})\pi_r(\underline{k}') + \omega_p^2 \left(\frac{1}{\underline{k}^2}\right) \underline{k}.\underline{\xi}(\underline{k})\underline{k}.\underline{\xi}(\underline{k}') \right\} \quad , \tag{2.17}$$

and that

$$H^{(3)} = -L^{(3)} \quad , \quad H^{(4)} = -L^{(4)} \quad , \text{ etc.} \tag{2.18}$$

III. LINEAR THEORY

The linearized equation of motion, which is derivable from (2.12),

$$\ddot{\xi}_r(\underline{k}) + \omega_p^2 \left(\frac{1}{k^2}\right) k_r k_s \xi_s(\underline{k}) = 0 \quad , \tag{3.1}$$

has solutions which fall into two categories. There are longitudinal modes for which $\underline{\xi}$ is parallel to $\underline{k}$, the frequency of which is ω_p; these are the plasma oscillations. The other modes are transverse, $\underline{\xi}$ being perpendicular to $\underline{k}$, and have zero frequency; these represent possible states of divergence-free shear motion of the electron gas. Since the "shear waves" are of zero frequency, it follows from the action-transfer relations[1+] that they can neither lose energy to nor gain energy from the plasma oscillations. These modes will affect each other only by modifying the dispersion relations. Since energy exchange among waves is a more important process, we shall neglect shear waves in what follows.

On noting that the reality of $\underline{\xi}(\underline{x},t)$ entails the fact that

$$\xi_r(-\underline{k},t) = \xi_r^*(\underline{k},t) \quad , \tag{3.2}$$

we see that the longitudinal modes are expressible as

$$\xi_r(\underline{k},t) = k_r \left\{ A(\underline{k}) e^{i\left(\omega_p t + \alpha(\underline{k})\right)} - A(-\underline{k}) e^{-i\left(\omega_p t + \alpha(-\underline{k})\right)} \right\} \quad . \tag{3.3}$$

The corresponding formula for the momentum is

$$\pi_r(\underline{k},t) = - i\omega_p k_r \left\{ A(-\underline{k}) e^{i\left(\omega_p t + \alpha(-\underline{k})\right)} + A(\underline{k}) e^{-i\left(\omega_p t + \alpha(\underline{k})\right)} \right\} \quad . \tag{3.4}$$

It is convenient to work with complex amplitudes as independent variables, but it is also necessary for subsequent purposes to maintain the canonical formalism. Both requirements can be met as follows: It is possible to transform to action angle variables $J(\underline{k})$, $\alpha(\underline{k})$, where $\alpha(\underline{k})$ has already been introduced and the appropriate form of $J(\underline{k})$ is found to be

$$J(\underline{k}) = 2\omega_p k^2 A^2(\underline{k}) \tag{3.5}$$

We may now form a canonical set of complex amplitudes by the formulas

$$a(\underline{k},t) = J^{1/2}(\underline{k}) e^{i\alpha(\underline{k})} \quad , \quad a^\dagger(\underline{k},t) = - iJ^{1/2}(\underline{k}) e^{-i\alpha(\underline{k})} \quad , \tag{3.6}$$

in which $a^\dagger(\underline{k},t)$ is to be regarded as the variable canonically conjugate to $a(\underline{k},t)$. The required expressions for the normal modes are now found to be

$$\xi_r(\underline{k},t) = (2\omega_p k^2)^{-1/2} k_r \left\{ a(\underline{k},t) e^{i\omega_p t} - i a^\dagger(-\underline{k},t) e^{-i\omega_p t} \right\} \quad , \tag{3.7}$$

and

$$\pi_r(\underline{k},t) = \left(\frac{\omega_p}{2k^2}\right)^{1/2} k_r \left\{ - i a(-\underline{k},t) e^{i\omega_p t} + a^\dagger(\underline{k},t) e^{-i\omega_p t} \right\} \quad . \tag{3.8}$$

Although, in the linear theory, a and $a^\dagger$ are time-independent, the above transformation will be used in nonlinear theory so that these variables will be time-dependent. We note, for later purposes, the obvious relation

$$a^\dagger(\underline{k},t) = - i a^*(\underline{k},t) \quad . \tag{3.9}$$

IV. NONLINEAR ANALYSIS BY THE DERIVATIVE-EXPANSION TECHNIQUE

In this section, we compute the dominant effect of the nonlinear terms $H^{(3)}$ and $H^{(4)}$ by using the derivative-expansion technique of I. We find from (2.13), (2.14), (2.18) and (3.7) that the cubic and quartic terms of the Hamiltonian are expressible as

$$H^{(3)} = i \sum_{\mathbf{k}+\mathbf{k}'+\mathbf{k}''=0} A(\mathbf{k},\mathbf{k}',\mathbf{k}'')\left(a(-\mathbf{k})e^{i\omega_p t} - ia^\dagger(\mathbf{k})e^{-i\omega_p t}\right)$$
$$\times\left(a(-\mathbf{k}')e^{i\omega_p t} - ia^\dagger(\mathbf{k}')e^{-i\omega_p t}\right)\left(a(-\mathbf{k}'')e^{i\omega_p t} - ia^\dagger(\mathbf{k}'')e^{-i\omega_p t}\right), \tag{4.1}$$

$$H^{(4)} = \sum_{\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''=0} B(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}''')\left(a(-\mathbf{k})e^{i\omega_p t} - ia^\dagger(\mathbf{k})e^{-i\omega_p t}\right)$$
$$\times\left(a(-\mathbf{k}')e^{i\omega_p t} - ia^\dagger(\mathbf{k}')e^{-i\omega_p t}\right)\left(a(-\mathbf{k}'')e^{i\omega_p t} - ia^\dagger(\mathbf{k}'')e^{-i\omega_p t}\right)$$
$$\times\left(a(-\mathbf{k}''')e^{i\omega_p t} - ia^\dagger(\mathbf{k}''')e^{-i\omega_p t}\right), \tag{4.2}$$

where

$$A(\mathbf{k},\mathbf{k}',\mathbf{k}'') = \frac{1}{12}\left(\frac{P}{2}\right)^{1/2}\left(\frac{1}{kk'k''}\right)(\mathbf{k}\cdot\mathbf{k}'\mathbf{k}\cdot\mathbf{k}'' + \mathbf{k}\cdot\mathbf{k}'\mathbf{k}'\cdot\mathbf{k}'' + \mathbf{k}\cdot\mathbf{k}''\mathbf{k}'\cdot\mathbf{k}''), \tag{4.3}$$

$$B(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}''') = -\frac{1}{96}\left(\frac{1}{kk'k''k'''}\right)$$
$$\times(\mathbf{k}\cdot\mathbf{k}'\mathbf{k}\cdot\mathbf{k}''\mathbf{k}\cdot\mathbf{k}''' + \mathbf{k}\cdot\mathbf{k}'\mathbf{k}'\cdot\mathbf{k}''\mathbf{k}'\cdot\mathbf{k}''' + \mathbf{k}\cdot\mathbf{k}''\mathbf{k}'\cdot\mathbf{k}''\mathbf{k}''\cdot\mathbf{k}''' + \mathbf{k}\cdot\mathbf{k}'''\mathbf{k}'\cdot\mathbf{k}'''\mathbf{k}''\cdot\mathbf{k}''')$$
$$+\frac{1}{12}\left(\frac{1}{kk'k''k'''}\right)\Big[|\mathbf{k}+\mathbf{k}'|^{-2}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}'(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}''(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}'''$$
$$+|\mathbf{k}+\mathbf{k}''|^{-2}(\mathbf{k}+\mathbf{k}'')\cdot\mathbf{k}(\mathbf{k}+\mathbf{k}'')\cdot\mathbf{k}'(\mathbf{k}+\mathbf{k}'')\cdot\mathbf{k}''(\mathbf{k}+\mathbf{k}'')\cdot\mathbf{k}'''$$
$$+|\mathbf{k}+\mathbf{k}'''|^{-2}(\mathbf{k}+\mathbf{k}''')\cdot\mathbf{k}(\mathbf{k}+\mathbf{k}''')\cdot\mathbf{k}'(\mathbf{k}+\mathbf{k}''')\cdot\mathbf{k}''(\mathbf{k}+\mathbf{k}''')\cdot\mathbf{k}'''$$
$$+|\mathbf{k}'+\mathbf{k}''|^{-2}(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{k}(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{k}'(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{k}''(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{k}'''$$
$$+|\mathbf{k}'+\mathbf{k}'''|^{-2}(\mathbf{k}'+\mathbf{k}''')\cdot\mathbf{k}(\mathbf{k}'+\mathbf{k}''')\cdot\mathbf{k}'(\mathbf{k}'+\mathbf{k}''')\cdot\mathbf{k}''(\mathbf{k}'+\mathbf{k}''')\cdot\mathbf{k}'''$$
$$+|\mathbf{k}''+\mathbf{k}'''|^{-2}(\mathbf{k}''+\mathbf{k}''')\cdot\mathbf{k}(\mathbf{k}''+\mathbf{k}''')\cdot\mathbf{k}'(\mathbf{k}''+\mathbf{k}''')\cdot\mathbf{k}''(\mathbf{k}''+\mathbf{k}''')\cdot\mathbf{k}'''\Big], \tag{4.4}$$

where we again drop the argument t from $a(\mathbf{k},t)$ for brevity. We now find from the canonical equations

$$\frac{da(\mathbf{k})}{dt} = \frac{\partial H}{\partial a^\dagger(\mathbf{k},t)}, \quad \frac{da^\dagger(\mathbf{k},t)}{dt} = -\frac{\partial H}{\partial a(\mathbf{k},t)}, \tag{4.5}$$

that, taking account of only the cubic and quartic terms, $a(\mathbf{k},t)$ satisfies the following differential equation:

$$\frac{da(\mathbf{k},t)}{dt} = 3\tilde{\omega}\sum_{\mathbf{k}+\mathbf{k}'+\mathbf{k}''=0} A(\mathbf{k},\mathbf{k}',\mathbf{k}'')e^{-i\omega_p t}$$
$$\times\left(a(-\mathbf{k}')e^{i\omega_p t} - a^*(\mathbf{k}')e^{-i\omega_p t}\right)\left(a(-\mathbf{k}'')e^{i\omega_p t} - a^*(\mathbf{k}'')e^{-i\omega_p t}\right)$$
$$-4\tilde{\omega}^2\sum_{\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''=0} B(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}''')e^{-i\omega_p t}\left(a(-\mathbf{k}')e^{i\omega_p t} - a^*(\mathbf{k}')e^{-i\omega_p t}\right)$$
$$\times\left(a(-\mathbf{k}'')e^{i\omega_p t} - a^*(\mathbf{k}'')e^{-i\omega_p t}\right)\left(a(-\mathbf{k}''')e^{i\omega_p t} - a^*(\mathbf{k}''')e^{-i\omega_p t}\right). \tag{4.6}$$

We have at this stage introduced for present convenience the parameter $\tilde{\omega}$ which will separate terms and effects according to the powers of their dependence upon amplitude. We may confirm this by noting that, if $\tilde{\omega}$ is initially absent from (4.6), it enters as indicated when $a(\mathbf{k},t)$ is replaced by $\tilde{\omega}a(\mathbf{k},t)$.

We now seek an approximate solution of Eq. (4.6) as follows. The effect of nonlinearity upon $a(\mathbf{k},t)$ is of two types: the introduction of harmonics, and slow time-variation of amplitude. We allow for these two effects by replacing a by the expansion

$$a \to a + \tilde{\omega}a^{I} + \tilde{\omega}^2 a^{II} + \dots, \tag{4.7}$$

and by a similar expansion of the contributions to the time-derivative of any quantity:

$$\frac{d}{dt} \to \frac{d}{dt} + \tilde{\omega}\frac{d^{I}}{dt} + \tilde{\omega}^2\frac{d^{II}}{dt} + \dots, \tag{4.8}$$

We see that the right-hand side of (4.6) contains no zero-frequency term to order $\tilde{\omega}$ so that

$$\frac{d^I a}{dt} = 0 \quad . \tag{4.9}$$

On examining the frequency-dependence of terms of order $\tilde{\omega}$ in (4.6), we see that a^I may be analyzed as follows:

$$a^I = a_1^I e^{i\omega_p t} + a_{-1}^I e^{-i\omega_p t} + a_{-3}^I e^{-3i\omega_p t} \quad , \tag{4.10}$$

of which a_1^I and a_{-3}^I will contribute to the principal second harmonic content, and a_{-1}^I will contribute to the principal zero-frequency content of the electrostatic waves.

On making the substitutions (4.7), (4.8) and using (4.9), (4.10) in (4.6), and separating terms of order $\tilde{\omega}$, we find that

$$\left.\begin{aligned} a_1^I(\underline{k}) &= -3i\omega_p^{-1} \sum_{\underline{k}+\underline{k}'+\underline{k}''=0} A(\underline{k},\underline{k}',\underline{k}'')\, a(-\underline{k}')a(-\underline{k}'') \quad , \\ a_{-1}^I(\underline{k}) &= -6i\omega_p^{-1} \sum_{\underline{k}+\underline{k}'+\underline{k}''=0} A(\underline{k},\underline{k}',\underline{k}'')\, a^*(\underline{k}')a(-\underline{k}'') \quad , \\ a_{-3}^I(\underline{k}) &= \quad i\omega_p^{-1} \sum_{\underline{k}+\underline{k}'+\underline{k}''=0} A(\underline{k},\underline{k}',\underline{k}'')\, a^*(\underline{k}')a^*(\underline{k}'') \quad . \end{aligned}\right\} \tag{4.11}$$

We now consider terms of (4.6) which are of order $\tilde{\omega}^2$, and remember that a is to be replaced by (4.7) on the left-hand side and also on the right-hand side. We are interested only in the zero-frequency contribution of order $\tilde{\omega}^2$ and this is found to be

$$\frac{d^{II}a(\underline{k})}{dt} = 6 \sum_{\underline{k}+\underline{k}'+\underline{k}''=0} A(\underline{k},\underline{k}',\underline{k}'')\Big\{a_{-1}^I(-\underline{k}')a(-\underline{k}'') - a^*(\underline{k}')a_1^I(-\underline{k}'') - a_{-1}^{I*}(\underline{k}')a(-\underline{k}'')$$

$$+ a_{-3}^{I*}(\underline{k}')a^*(\underline{k}'')\Big\} + 1 \ldots \sum_{\underline{k}+\underline{k}'+\underline{k}''+\underline{k}'''=0} B(\underline{k},\underline{k}',\underline{k}'',\underline{k}''')a^*(\underline{k}')a(-\underline{k}'')a(-\underline{k}''') \quad . \tag{4.12}$$

On using formulas (4.11), we find that (4.12) may be rewritten as

$$\frac{d^{II}a(\underline{k}_1)}{dt} = i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}_3+\underline{k}_4} C(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)a^*(\underline{k}_2)a(\underline{k}_3)a(\underline{k}_4) \quad , \tag{4.13}$$

where

$$\begin{aligned} C(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4) = 12\omega_p^{-1} \Big\{ & A(\underline{k}_1,\underline{k}_2,-\underline{k}_1-\underline{k}_2) \;\; A(\underline{k}_3,\underline{k}_4,-\underline{k}_3-\underline{k}_4) \\ & - 3A(\underline{k}_1,-\underline{k}_4,\underline{k}_4-\underline{k}_1) \;\; A(\underline{k}_2,-\underline{k}_3,\underline{k}_3-\underline{k}_2) \\ & - 3A(\underline{k}_1,-\underline{k}_3,\underline{k}_3-\underline{k}_1) \;\; A(\underline{k}_2,-\underline{k}_4,\underline{k}_4-\underline{k}_2) \\ & + B(\underline{k}_1,\underline{k}_2,-\underline{k}_3,-\underline{k}_4)\Big\} \quad . \end{aligned} \tag{4.14}$$

Equation (4.13) represents the dominant energy-transfer process due to nonlinearity as a process which may be analyzed into the interactions of groups of four waves. The energy $\epsilon_{\underline{k}}$ to be attributed to each wave is proportional to $a^*(\underline{k})a(\underline{k})$. We now find, from the symmetry properties of the "kernel" $C(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)$,

$$C(\underline{k}_2,\underline{k}_1,\underline{k}_3,\underline{k}_4) = C(\underline{k}_1,\underline{k}_2,\underline{k}_4,\underline{k}_3) = C(\underline{k}_3,\underline{k}_4,\underline{k}_1,\underline{k}_2) = C(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4) \quad , \tag{4.15}$$

that

$$\frac{d\epsilon(\underline{k}_1)}{dt} = \frac{d\epsilon(\underline{k}_2)}{dt} = -\frac{d\epsilon(\underline{k}_3)}{dt} = -\frac{d\epsilon(\underline{k}_4)}{dt} \quad . \tag{4.16}$$

Hence energy is drawn in equal amounts from one pair of waves and distributed equally among the other pair of waves. Momentum is exchanged in the same way. Equation (4.16) agrees with the action-transfer relations[14] for four waves of equal frequency, the wave numbers of which are related by

$$\underline{k}_1 + \underline{k}_2 = \underline{k}_3 + \underline{k}_4 \quad . \tag{4.17}$$

On using (4.3) and (4.4), we find that

$$
\begin{aligned}
C(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4) = -\tfrac{1}{8}(k_1k_2k_3k_4)^{-1}\Big[&\underline{k}_1\cdot\underline{k}_2\,\underline{k}_3\cdot\underline{k}_4 + \underline{k}_1\cdot\underline{k}_3\,\underline{k}_2\cdot\underline{k}_4 \\
&+ \underline{k}_1\cdot\underline{k}_3\,\underline{k}_2\cdot\underline{k}_4 + \underline{k}_1\cdot\underline{k}_4\,\underline{k}_2\cdot\underline{k}_3\Big] \\
+ \tfrac{1}{16}(k_1k_2k_3k_4)^{-1}|\underline{k}_1+\underline{k}_2|^{-2}\Big[&\underline{k}_1\cdot(\underline{k}_1+\underline{k}_2)\,\underline{k}_2\cdot(\underline{k}_1+\underline{k}_2) \\
&\underline{k}_3\cdot(\underline{k}_1+\underline{k}_2)\,\underline{k}_4\cdot(\underline{k}_1+\underline{k}_2) \\
&+\left\{k_1^2k_2^2-(\underline{k}_1\cdot\underline{k}_2)^2\right\}\left\{k_3^2k_4^2-(\underline{k}_3\cdot\underline{k}_4)^2\right\}\Big] \\
+ \tfrac{1}{8}(k_1k_2k_3k_4)^{-1}|\underline{k}_1-\underline{k}_3|^{-2}\Big[&\underline{k}_1\cdot(\underline{k}_1-\underline{k}_3)\,\underline{k}_2\cdot(\underline{k}_1-\underline{k}_3) \\
&\underline{k}_3\cdot(\underline{k}_1-\underline{k}_3)\,\underline{k}_4\cdot(\underline{k}_1-\underline{k}_3) \\
&-\left\{k_1^2k_3^2-(\underline{k}_1\cdot\underline{k}_3)^2\right\}\left\{k_2^2k_4^2-(\underline{k}_2\cdot\underline{k}_4)^2\right\}\Big] \\
+ \tfrac{1}{8}(k_1k_2k_3k_4)^{-1}|\underline{k}_1-\underline{k}_4|^{-2}\Big[&\underline{k}_1\cdot(\underline{k}_1-\underline{k}_4)\,\underline{k}_2\cdot(\underline{k}_1-\underline{k}_4) \\
&\underline{k}_3\cdot(\underline{k}_1-\underline{k}_4)\,\underline{k}_4\cdot(\underline{k}_1-\underline{k}_4) \\
&-\left\{k_1^2k_4^2-(\underline{k}_1\cdot\underline{k}_4)^2\right\}\left\{k_2^2k_3^2-(\underline{k}_2\cdot\underline{k}_3)^2\right\}\Big]\,. \qquad (4.18)
\end{aligned}
$$

By further algebraic manipulations it is possible to write the formula in the form

$$
\begin{aligned}
C = \tfrac{1}{4}(k_1k_2k_3k_4)^{-1}\Big[&-\underline{k}_1\cdot\underline{k}_3\,\underline{k}_1\cdot\underline{k}_4\,\underline{k}_1\cdot(\underline{k}_1+\underline{k}_4) - \underline{k}_2\cdot\underline{k}_3\,\underline{k}_2\cdot\underline{k}_4\,\underline{k}_1\cdot(\underline{k}_1+\underline{k}_4)\Big] \\
+ \tfrac{1}{32}(k_1k_2k_3k_4)^{-1}|\underline{k}_1+\underline{k}_2|^{-2}\Big[&\underline{k}_1\cdot(\underline{k}_1+\underline{k}_2)\,\underline{k}_2\cdot(\underline{k}_1+\underline{k}_2)\,\underline{k}_3\cdot(\underline{k}_1+\underline{k}_2)\,\underline{k}_4\cdot(\underline{k}_1+\underline{k}_2) \\
&-\left\{k_1^2k_2^2-(\underline{k}_1\cdot\underline{k}_2)^2\right\}\left\{k_3^2k_4^2-(\underline{k}_3\cdot\underline{k}_4)^2\right\}\Big]\,. \qquad (4.19)
\end{aligned}
$$

which shows explicitly that there is no singularity at $\underline{k}_1 - \underline{k}_3 = \underline{k}_4 - \underline{k}_2 = 0$ or at $\underline{k}_1 - \underline{k}_4 = \underline{k}_3 - \underline{k}_2 = 0$. One may also verify that there is no singularity at $\underline{k}_1 + \underline{k}_2 = \underline{k}_3 + \underline{k}_4 = 0$.

V. NONLINEAR ANALYSIS BY CANONICAL TRANSFORMATION

In this section we use a technique alternative to that of Section IV for calculating the dominant wave-interaction process due to nonlinearity of the equations describing electrostatic waves. If $H^{(3)}$ contains a term of zero-frequency, this term would represent the dominant wave interaction. Since $H^{(4)}$ contains a zero-frequency term, this certainly contributes to the dominant interaction, but one must also consider the possibility that $H^{(3)}$ gives rise to a wave-interaction process comparable in magnitude with that due to $H^{(4)}$. The processes to be considered are characterized by the "Feynmann diagrams" shown in Fig. 1.

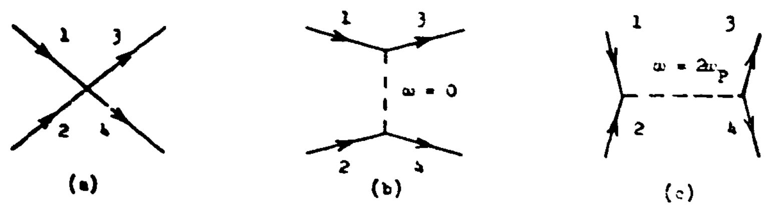

Figure 1

Figure 1a represents the direct interaction of four waves associated with the zero-frequency part of $H^{(4)}$. Figures 1b and 1c represent competing processes associated with $H^{(3)}$, which involve the excitation of "virtual waves" of frequencies zero and $2\omega_p$, respectively.

Since we wish to calculate the wave-interaction process only to order $H^{(4)}$, we may proceed as follows. We carry out a canonical transformation of the dynamical variables which eliminates the cubic contribution to the Hamiltonian, and then pick out the zero-frequency contribution to the resulting quartic part of the Hamiltonian. The technique for finding this transformation is given in the Appendix.

We apply the method of the Appendix to the present problem by identifying the initial set of variables with the complex variables $a^\dagger(\underline{k})$, $a(\underline{k})$ and identifying h^{I}, h^{II} with $H^{(3)}$, $H^{(4)}$ as given by (4.1) and (4.2). We write the transformed variables as $b^\dagger(\underline{k})$, $b(\underline{k})$ and the transformed Hamiltonian as $\tilde{H}$. Then we see from the Appendix that the generating function $U^{I}\left(b^\dagger(\underline{k}),(\underline{k}),t\right)$ is defined by the equation

$$\frac{\partial U^{I}}{dt} = - H^{(3)} \quad , \tag{5.1}$$

in which the arguments $a^\dagger$, a of $H^{(3)}$ are replaced by $b^\dagger$, b. We find that

$$U^{I} = \omega_p^{-1} \sum_{\underline{k}+\underline{k}'+\underline{k}''=0} A(\underline{k},\underline{k}',\underline{k}'') \left[-\frac{1}{3} b(-\underline{k})b(-\underline{k}')b(-\underline{k}'')e^{3i\omega_p t} + 3ib^\dagger(\underline{k})b(-\underline{k}')b(-\underline{k}'')e^{i\omega_p t} - 3b^\dagger(\underline{k})b^\dagger(\underline{k}')b(-\underline{k}'')e^{-i\omega_p t} + \frac{1}{3} ib^\dagger(\underline{k})b^\dagger(\underline{k}')b^\dagger(\underline{k}'')e^{-3i\omega_p t}\right] . \tag{5.2}$$

On noting that

$$A(-\underline{k},-\underline{k}',-\underline{k}'') = A(\underline{k},\underline{k}',\underline{k}'') \quad , \tag{5.3}$$

we find from (A.9) that

$$b^{\dagger I}(\underline{k}) = \omega_p^{-1} \sum_{\underline{k}+\underline{k}'+\underline{k}''=0} A(\underline{k},\underline{k}',\underline{k}'') \left[- b(\underline{k}')b(\underline{k}'')e^{3i\omega_p t} + 6ib^\dagger(-\underline{k}')b(\underline{k}'')e^{i\omega_p t} - 3b(-\underline{k}')b^\dagger(-\underline{k}'')e^{-i\omega_p t}\right] , \tag{5.4}$$

$$b^{I}(k) = \omega_p^{-1} \sum_{\underline{k}+\underline{k}'+\underline{k}''=0} A(\underline{k},\underline{k}',\underline{k}'') \left[- 3ib(-\underline{k}')b(-\underline{k}'')e^{i\omega_p t} + 6b^\dagger(\underline{k}')b(-\underline{k}'')e^{-i\omega_p t} - ib^\dagger(\underline{k}')b^\dagger(\underline{k}'')e^{-3i\omega_p t}\right] \tag{5.5}$$

We may now find the function $\tilde{H}^{(4)}$ by Eq. (A.12). However, on noting that we shall need only the zero frequency part of this function, we see that the term involving the arbitrary coefficient θ drops out

of the expression, leaving

$$\tilde{H}^{(4)}_{ZF} = H^{(4)}_{ZF} + \frac{i}{2}\sum_{\underline{k}}\left[b^{*I}(\underline{k})\frac{\partial b^{I}(\underline{k})}{\partial t} - b^{I}(\underline{k})\frac{\partial b^{*I}(\underline{k})}{\partial t}\right]_{ZF} \quad , \qquad (5.6)$$

where the subscript ZF refers to the zero-frequency part of a function. Evaluating (5.6) by means of (4.3), (4.4), (5.4) and (5.5), we find that

$$\tilde{H}^{(4)}_{ZF} = -\frac{1}{2}\sum_{\underline{k}_1+\underline{k}_2=\underline{k}_3+\underline{k}_4} C(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)b^*(\underline{k}_1)b^*(\underline{k}_2)b(\underline{k}_3)b(\underline{k}_4) \quad , \qquad (5.7)$$

where $C(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)$ is identical with the function (4.14). We see from (4.5) that the canonical equation derivable from (5.7) is

$$\frac{db(\underline{k}_1)}{dt} = i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}_3+\underline{k}_4} C(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)b^*(\underline{k}_2)b(\underline{k}_3)b(\underline{k}_4) \quad , \qquad (5.9)$$

which is identical in form with (4.13). Hence the derivative-expansion technique and the canonical-transformation technique lead to the same formula for the dominant nonlinear wave interaction.

VI. DISCUSSION

We remark first of all that the interaction kernel (4.19) vanishes identically if the spectrum is one-dimensional. Hence there is no conflict between the present theory and the known properties of one-dimensional disturbances of cold electron plasmas.[2] As shown in I, the class of interactions represented by (4.13) may be divided into two groups, "coherent" interactions for which $\underline{k}_1 = \underline{k}_3$ or $\underline{k}_4$ and, in consequence, $\underline{k}_2 = \underline{k}_4$ or $\underline{k}_3$, and "incoherent" interactions for which these equalities do not hold. Coherent interactions give rise only to dispersion, whereas incoherent interactions are responsible for energy exchange between waves and hence for spectral decay. The former effect is much simpler and will be treated here; discussion of the latter effect is postponed for a subsequent article.

On separating out the coherent interactions, (4.13) takes the form

$$\frac{d^{II}a(\underline{k})}{dt} = i\sum_{\underline{k}'} C(\underline{k},\underline{k}')a^*(\underline{k}')a(\underline{k}')a(\underline{k}) \quad , \qquad (6.1)$$

where

$$C(\underline{k},\underline{k}') = C(\underline{k},\underline{k}',\underline{k},\underline{k}') \quad . \qquad (6.2)$$

We find from (4.19) that

$$C(\underline{k},\underline{k}') = \frac{1}{6}\frac{\left\{k^2k'^2 - (\underline{k}\cdot\underline{k}')^2\right\}^2}{k^2k'^2|\underline{k}+\underline{k}'|^2} \quad . \qquad (6.3)$$

We see from (6.1) that this equation gives rise to a frequency change given by the dispersion relation

$$\omega = \omega_p + \Delta\Omega(\underline{k}) \quad , \qquad (6.4)$$

where

$$\Delta\Omega(\underline{k}) = \sum_{\underline{k}'} C(\underline{k},\underline{k}')|a(\underline{k}')|^2 \quad . \qquad (6.5)$$

We now assume that the wave under consideration, a "test" wave, is interacting with a large number of "background" waves, and that the great majority of background waves are of much shorter wave-length than the test wave, i.e. $k \ll k'$. With this approximation, (6.3) takes the approximate form

$$C(\underline{k},\underline{k}') = \frac{1}{6}\,\frac{k^4 k'^4 - 2k^2 k'^2 (\underline{k}\cdot\underline{k}')^2 + (\underline{k}\cdot\underline{k}')^4}{k^2 k'^4} \quad . \tag{6.6}$$

We next assume that background waves are excited isotropically and so replace $C(\underline{k},\underline{k}')$ in (6.5) by the value one obtains by averaging over all directions of the vector $\underline{k}'$, which is found to be

$$\left\langle C(\underline{k},\underline{k}')\right\rangle_{\underline{k}'} = \frac{4}{45}\,k^2 \quad . \tag{6.7}$$

We next note that in consequence of the definition (3.6) and the relation (2.11), the energy density $\epsilon(\underline{k})$ of a wave is related to its amplitude by

$$\epsilon(\underline{k}) = \omega_p m n\, |a(\underline{k})|^2 \quad . \tag{6.8}$$

Hence Eq. (6.5) becomes

$$\Delta\omega(\underline{k}) = \frac{4}{45}\,\frac{\mathcal{E}}{\omega_p m n}\,k^2 \quad . \tag{6.9}$$

where $\mathcal{E}$ is the energy density of the background waves. Since half the energy of electrostatic waves is kinetic and half electrostatic, we see that

$$\mathcal{E} = mn\langle v^2\rangle \quad , \tag{6.10}$$

where $\langle v^2\rangle$ is the mean square speed of electrons due to the background excitation. Hence (6.4) finally becomes

$$\omega = \omega_p + \frac{4}{45}\langle v^2\rangle k^2 \quad . \tag{6.11}$$

which is of the same form as the dispersion relation describing electrostatic waves in a plasma of nonzero temperature in the small-wave-number approximation.[15] It is at present not clear to the author whether the frequency shift (6.11) is additional to that contained in the familiar dispersion relation, or whether this method of calculation of the frequency shift in a "thermal" plasma is an alternative equivalent to the standard procedure; in the latter case the difference between the factor 1/2 which appears in the standard formula and 4/45 which occurs in (6.11) must be attributed to transverse (divergence-free) waves which we have neglected in the present article. The discussion of the interaction of plasma oscillations and transverse waves given in I indicates that this effect is inadequate to account for the difference. This strongly suggests that the effect which we have here evaluated is additional to that given in the standard dispersion relation.

APPENDIX

<u>Canonical Transformation for Reduction of Perturbation Hamiltonian</u>

Consider a dynamical system described by dynamical variables p_r, q_r, and a Hamiltonian $h(p_r, q_s, t)$ which is given as a perturbation expansion in a parameter $\tilde{\omega}$ of the form

$$h = \tilde{\omega} h^I + \tilde{\omega}^2 h^{II} + \dots \quad . \tag{A.1}$$

Since there is no zero-order term, p_r, q_r are constants of the motion in zero-order theory.

We consider a canonical transformation of the system to new variables P_r, Q_r and a new Hamiltonian $H(P_r, Q_s, t)$. We consider the transformation of variables to be expressible in the form

$$\left.\begin{aligned} p_r &= P_r - \tilde{\omega} P_r^I + \tilde{\omega}^2 P_r^{II} + \dots \quad , \\ q_r &= Q_r - \tilde{\omega} Q_r^I + \tilde{\omega}^2 Q_r^{II} + \dots \quad , \end{aligned}\right\} \tag{A.2}$$

wherein P_r^I etc. are to be expressed as functions of P_r, Q_r, t. The transformation is then the identity transformation in zero-order theory. The new Hamiltonian will in general be expressible as

$$H = \tilde{\omega} H^I + \tilde{\omega}^2 H^{II} + \dots \quad . \tag{A.3}$$

We wish to find a canonical transformation of the above form for which $H^I = 0$.

The essential properties of the original variables and Hamiltonian are summarized by the relation

$$\delta h = \dot{q}_r \delta p_r - \dot{p}_r \delta q_r \quad . \tag{A.4}$$

The term h may be replaced by the expression (A.1) and the argument of h^I etc. replaced by the expressions (A.2). The right-hand side of (A.4) may also be expressed in terms of P_r, Q_r by means of (A.2). If the resulting relation reduces to

$$\delta H = \dot{Q}_r \delta P_r - \dot{P}_r \delta Q_r \quad . \tag{A.5}$$

then the transformation represented by (A.2) is canonical.

On making the above substitutions, (A.4) takes the form

$$\tilde{\omega}\left[\frac{\partial h^I}{\partial P_r}\delta P_r + \frac{\partial h^I}{\partial Q_r}\delta Q_r\right] + \tilde{\omega}^2\delta\left[P_r^I\frac{\partial h^I}{\partial P_r} + Q_r^I\frac{\partial h^I}{\partial Q_r}\right] + \tilde{\omega}^2\left[\frac{\partial h^{II}}{\partial P_r}\delta P_r + \frac{\partial h^{II}}{\partial Q_r}\delta Q_r\right] + \dots$$

$$= \left[\dot{Q}_r + \tilde{\omega}\left(\frac{\partial Q_r^I}{\partial P_s}\dot{P}_s + \frac{\partial Q_r^I}{\partial Q_s}\dot{Q}_s + \frac{\partial Q_r^I}{\partial t}\right) + \tilde{\omega}^2\left(\frac{\partial Q_r^{II}}{\partial P_s}\dot{P}_s + \frac{\partial Q_r^{II}}{\partial Q_s}\dot{Q}_s + \frac{\partial Q_r^{II}}{\partial t}\right) + \dots\right]$$

$$\times\left[\delta P_r + \tilde{\omega}\left(\frac{\partial P_r^I}{\partial P_t}\delta P_t + \frac{\partial P_r^I}{\partial Q_t}\delta Q_t\right) + \tilde{\omega}^2\left(\frac{\partial P_r^{II}}{\partial P_t}\delta P_t + \frac{\partial P_r^{II}}{\partial Q_t}\delta Q_t\right) + \dots\right]$$

$$- \left[\dot{P}_r + \tilde{\omega}\left(\frac{\partial P_r^I}{\partial P_s}\dot{P}_s + \frac{\partial P_r^I}{\partial Q_s}\dot{Q}_s + \frac{\partial P_r^I}{\partial t}\right) + \tilde{\omega}^2\left(\frac{\partial P_r^{II}}{\partial P_s}\dot{P}_s + \frac{\partial P_r^{II}}{\partial Q_s}\dot{Q}_s + \frac{\partial P_r^{II}}{\partial t}\right) + \dots\right]$$

$$\times\left[\delta Q_r + \tilde{\omega}\left(\frac{\partial Q_r^I}{\partial P_t}\delta P_t + \frac{\partial Q_r^I}{\partial Q_t}\delta Q_t\right) + \tilde{\omega}^2\left(\frac{\partial Q_r^{II}}{\partial P_t}\delta P_t + \frac{\partial Q_r^{II}}{\partial Q_t}\delta Q_t\right) + \dots\right] \quad . \tag{A.6}$$

We now replace $\dot{Q}_r\delta P_r - \dot{P}_r\delta Q_r$ on the right-hand side of (A.6) by the variation of the expression (A.3) expressing this variation in terms of δP_r, δQ_r. If we now separate terms of first order in $\tilde{\omega}$, we find that (A.6) is satisfied to this order and that we may satisfy the requirement

$$H^I = 0 \tag{A.7}$$

by introducing a generating function $U^I(P_r, Q_s, t)$ which is to satisfy

$$\frac{\partial U^I}{\partial t} = - h^I \tag{A.8}$$

and from which P_r^I, Q_r^I are obtained by means of the equations

$$P_r^I = \frac{\partial U^I}{\partial Q_r} \quad , \quad Q_r^I = -\frac{\partial U^I}{\partial P_r} \quad . \tag{A.9}$$

We now wish to find second order terms of (A.2) and (A.3) which are consistent with (A.7). We first note from (A.8) and (A.9) that

$$\frac{\partial h^I}{\partial P_r} = \frac{\partial Q_r^{\,I}}{\partial t} \quad , \quad \frac{\partial h^I}{\partial Q_r} = -\frac{\partial P_r^{\,I}}{\partial t} \quad , \tag{A.10}$$

and so express the second term of the left-hand side of (A.6) in terms of δP_r , δQ_r . If we now separate terms of (A.6) of order $\tilde{\omega}^2$, we find that (A.6) is satisfied to this order if P_r^{II} and Q_r^{II} are given by

$$\left.\begin{aligned} P_r^{II} &= (-\tfrac{1}{2} + \theta) P_s^{\,I} \frac{\partial Q_s^{\,I}}{\partial Q_r} + (\tfrac{1}{2} + \theta) Q_s^{\,I} \frac{\partial P_s^{\,I}}{\partial Q_r} \quad , \\ Q_r^{II} &= (\tfrac{1}{2} - \theta) P_s^{\,I} \frac{\partial Q_s^{\,I}}{\partial P_r} + (-\tfrac{1}{2} - \theta) Q_s^{\,I} \frac{\partial P_s^{\,I}}{\partial P_r} \quad , \end{aligned}\right\} \tag{A.11}$$

and H^{II} is given by

$$H^{II} = h^{II} + (\tfrac{1}{2} + \theta) P_r^{\,I} \frac{\partial Q_r^{\,I}}{\partial t} + (-\tfrac{1}{2} + \theta) Q_r^{\,I} \frac{\partial P_r^{\,I}}{\partial t} \quad , \tag{A.12}$$

where θ may be chosen arbitrarily.

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